Homework Assignment #2

Reading
Lecture 6 — Volume 1: 3.8-3.8.5
Lecture 7 — Volume 1: 4.1-4.3.2.3 4.4-4.5.1.2
Lecture 8 — Volume 1: 4.7-4.7.1.2

Announcements

Proposals for the Experimental Projects are due on Lecture 5 Day. Laboratory partners should submit a joint proposal that contains a brief statement of the hypothesis that you propose to test and the method that you will use to test it. The proposal should be on a single sheet of paper.

No other form of submission will be accepted.

Other information about the proposal can be found in the Laboratory Manual, which was distributed in Lecture 2 and is available from our home page.

The deadline for signing up for a laboratory session is also on Lecture 5 Day. You can sign up for a laboratory session electronically, by clicking “Signup for a laboratory session” on our home page.

Exercise 1. Two time constants are involved in two-compartment diffusion through a membrane: the steady-state time constant of the membrane ($\tau_{ss}$) and the equilibrium time constant for the two compartments ($\tau_{eq}$). Without the use of equations, describe these two time constants.

Exercise 2. Two solutions are separated by a membrane of thickness $d$ and surface area $A$, as shown in Figure 1. The membrane:solution partition coefficient $k_n = 2$. Bath #1 has an infinite

![Figure 1: Arrangements of baths and membrane.](image-url)
volume and has a concentration of solute $n$ of $C$. Bath #2 has volume $V_2$ and a concentration of solute $n$ that is $C/4$.

(a) Assume that the membrane has reached steady state at $t = 0$.

(a1) Let $c_n(x, t)$ represent the concentration of solute $n$ at position $x$ and time $t$. Sketch $c_n(x, 0)$ for values of $x$ that include the membrane and the two baths.

(a2) Is the system in equilibrium at $t = 0$? Explain.

(b) Let $P_n$ represent the steady state permeability of the membrane to solute $n$.

(b1) Assume that the membrane can be treated as a thin membrane and that the baths are “well stirred.” The concentration of solute $n$ in Bath #2 $c^2_n(t) = C/4$ for $t < 0$. Sketch $c^2_n(t)$ for $t > 0$.

(b2) Assume that the membrane is too thick to be treated as a thin membrane. Describe how the plot of $c^2_n(t)$ versus $t$ would differ from the plot in (b1).

Exercise 3. The time course of one-dimensional diffusion of a solute from a point source in space and time has the form

$$c_n(x, t) = \frac{n_o}{\sqrt{4\pi Dt}} e^{-x^2/4Dt},$$

where $n_o$ is the number of moles of solute per unit area placed at $x = 0$ at $t = 0$. $c_n(x, t)$ is computed for locations $x_a$ and $x_b$ as shown in Figure 2. Is $x_a > x_b$ or is $x_a < x_b$? Explain.

![Figure 2: Concentration of solute $n$ as a function of time for two locations $x_a$ and $x_b$.](image)

Exercise 4. Cell $a$ and cell $b$ have identical dimensions but different permeabilities for solute $n$, $P^n_a$ and $P^n_b$, respectively. The cells are placed in identical solutions that contain the permeant solute $n$. The intracellular concentrations for cell $a$ and for cell $b$ are shown in Figure 3. Is $P^n_a > P^n_b$ or is $P^n_a < P^n_b$? Explain.

![Figure 3: Concentration of solute $n$ as a function of time in two cells, $c_n^a(t)$ and $c_n^b(t)$ that have different permeabilities for solute $n$.](image)
**Problem 1.** Two solutions of an uncharged solute $S$ have volumes $V_1 = 100 \text{ cm}^3$ and $V_2 = 50 \text{ cm}^3$ and are separated by a thin membrane (area $A = 25 \text{ cm}^2$) permeant to $S$ and impermeant to water.

The flux of $S$ through the membrane obeys Fick’s law for membranes. At time $t = 0$, the concentration of $S$ in solution 1 is $c_1(0) = 100 \text{ mol/m}^3$. The initial concentration of $S$ in solution 2 is not known. The flux of $S$ through the membrane in the positive $x$ direction is found to be an exponential function of time as shown in the plot.

(a.) Determine the concentration $c_1(t)$ of $S$ in solution 1 and the concentration $c_2(t)$ of $S$ in solution 2 as functions of time, assuming that the solutions are well-stirred. Sketch $c_1(t)$ and $c_2(t)$ on suitably labeled axes.

(b.) Determine numerical values for the final concentrations of $S$: $c_1(\infty)$ and $c_2(\infty)$. If it is not possible to determine numerical values, list the other information that would be needed to determine a numerical value.

**Problem 2.** A thin membrane and a thick membrane, that are otherwise identical, are used to separate identical solutions of volume $V = 1 \text{ cm}^3$ (Figure 4). All the membrane surfaces facing the solutions have area $A = 1 \text{ cm}^2$. The thin membrane has thickness $d_s = 10^{-4} \text{ cm}$; the thick membrane has thickness $d_l = 1 \text{ cm}$. For $t < 0$ the aqueous solutions on both sides of the membrane are identical and do not contain solute $n$. At $t = 0$ a small concentration of solute $n$ is added to the solution on side 1 of the membrane. You may assume that there is no water flow across the membrane. The diffusion coefficient and membrane:solution partition coefficient of $n$ in both membranes are $D_n = 10^{-5} \text{ cm}^2/\text{s}$ and $k_n = 2$, respectively.

![Diagram of thin and thick membranes](image)

Figure 4: Schematic diagrams of thin and thick membranes.

a) For each membrane, estimate the time, $\tau_{ss}$, it takes for the concentration profile in the membrane to reach its steady-state spatial distribution.

b) For each membrane, find the time, $\tau_{eq}$, it takes for the solutions on the two sides of the membrane to come to equilibrium assuming that the spatial distribution of solute in the membrane is the steady-state distribution.
c) Is it reasonable to assume that Fick’s Law for membranes applies for the thin membrane at each instant in time, i.e., does

\[ \phi_n(t) = P_s (c_n^1(t) - c_n^2(t)) \]

\[ = \frac{P_s}{d} \frac{d}{dx} (c_n^1(t) - c_n^2(t)) \]

d) Is it reasonable to assume that Fick’s Law for membranes applies for the thick membrane at each instant in time, i.e., does

\[ \phi_n(t) = P_l (c_n^1(t) - c_n^2(t)) \]

\[ = \frac{P_l}{d} \frac{d}{dx} (c_n^1(t) - c_n^2(t)) \]

\[ \text{where} \quad P_s \text{ and } P_l \text{ are the permeabilities of the thin and thick membranes, respectively.} \]

**Problem 3.** Two compartments are separated by a membrane of thickness \( d \) as shown in the following figure.

The compartments have equal cross-sectional areas, but their lengths may differ so that the volume of compartment 1, \( V_1 \), need not be equal to the volume of compartment 2, \( V_2 \). The compartments are filled with aqueous solutions that contain a single solute \( A \), and the concentration of solute within each compartment is kept uniform by stirring. The diffusion coefficient for \( A \) in the membrane is \( D > 0 \). The membrane:solution partition coefficient for \( A \) is \( k \geq 0 \). The membrane is impermeable to water. External sources deliver a flux \( \phi_1 \) of solute \( A \) into compartment 1 and remove a flux \( \phi_2 \) of solute \( A \) from compartment 2. The fluxes \( \phi_1 \) and \( \phi_2 \) are constant in time. Assume that the membrane volume is negligible compared to the volumes of the two compartments.

Let \( c(x, t) \) represent the solute concentration at location \( x \) and time \( t \). The initial concentration profile \( c(x, 0) \) is given in the previous figure. The following plots show 12 possible final concentration profiles that could result as \( t \to \infty \).
Identify which of the 12 nal concentration profiles could result for the special conditions described in each of the following parts. Record all possible profile numbers for each part or record none if none applies.

(a.) $\phi_1 = \phi_2 = 0$ and $V_1 = V_2$ and $k = 0$.

(b.) $\phi_1 = \phi_2 = 0$ and $V_1 = V_2$ and $k = 1$.

(c.) $\phi_1 = \phi_2 = 0$ and $V_1 = V_2$ and $k > 1$.

(d.) $\phi_1 = \phi_2 = 0$ and $V_2 = 5V_1$ and $k > 1$.

(e.) $\phi_1 > \phi_2 > 0$ and $V_1 = V_2$ and $k = 1$.

(f.) $\phi_1 > \phi_2 > 0$ and $V_1 = V_2$ and $k = 1$.

(g.) $\phi_1 > \phi_2 > 0$ and $V_2 = 5V_1$ and $k = 1$.

**Problem 4.** An experiment is performed to determine the permeability, $P_X$, of the membrane of a cell to solute $X$. The cell is spherical and has a radius of 72 $\mu$m. It is placed in a solution containing solute $X$ for a sufficient time to load the cell with $N_X$ moles of $X$. A set of identical vials containing identical solutions that do not contain the solute $X$ are prepared. The cell is then immersed successively in the series of these vials for $T = 10$ minutes per vial, i.e., 10 minutes in vial 1, followed by 10 minutes in vial 2, followed by 10 minutes in vial 3, etc., as shown in the following figure.
The number of moles of solute \( X \) in vial \( k \), is \( n_X(k) \). Assume that the volume of the cell is constant and negligible compared with the volume of a vial, and hence, the concentration of solute in a vial is always negligible compared to that in the cell. The solute permeates the membrane according to Fick’s Law for membranes.

(a.) Determine an expression for the number of moles of \( X \) in the cell as a function of time, \( n_X(t) \). You may write this expression in terms of literals such as \( N_X \) and a suitably defined time constant. Assume that the transfer of the cell from vial to vial takes no time.

(b.) Determine an expression for the total quantity of \( X \) in the \( k^{th} \) vial, \( n_X(k) \) in terms of literals such as \( N_X \), \( T \), and a suitably defined time constant.

(c.) For the measurements shown in figure shown below, determine the numerical values of \( P_X \) and \( N_X \).

**Problem 5.** Two-compartment diffusion is based on 4 assumptions:

1. The two compartments are well-mixed so that the concentrations of solute \( n \) are uniform and have values at time \( t \) of \( c^1_n(t) \) and \( c^2_n(t) \).
2. Solute particles are conserved, e.g., there is no chemical reaction present that either creates or destroys particles.
3. The membrane is sufficiently thin that the number of solute particles contained in the membrane at any time is negligibly small.
4. The membrane is sufficiently thin that at each instant in time the concentration profile in the membrane is in steady state.
In this problem, the course software will be used to determine conditions for the validity of assumptions 3 and 4. (The course software is available on Athena. Select 6.021J from Dash. Use the Matlab® 5.3 version, which can also be obtained from the subject web site. Select Macroscopic Diffusion.) Specifically, the software will be used to determine the effect of bath dimensions on two-compartment diffusion without making these two assumptions.

In all parts of the problem, use the Two compartments option of the software (select the rightmost “pro le” in the “MD initial condition” window). Set the membrane width to 0.01 cm, and the concentration of the left bath to 70 and right bath to 10 mol/cm$^3$. Leave the drift velocity and reaction rate at 0 and the diffusion coef cient at $10^{-5}$ cm$^2$/s. Select the “membrane view” so that both the membrane and a little of each bath are visible on the screen. Note that the left edge of the membrane is at a position of 0 cm. Draw some initial concentration in the membrane. Make sure all ordinate scales on all plots are 0 to 100 mol/cm$^3$. Keep these parameters x ed throughout this problem. To reduce the time taken for the computation, reduce the spatial resolution in the membrane to 10 points in space. The software will determine the time increment. Initially set the number of steps so that the end time is 5 s. Notice that you must press the “UPDATE” button on EACH window to have changes in that window propagate to the other windows.

For each of the pairs of bath widths answer the following questions.

<table>
<thead>
<tr>
<th>Left Bath</th>
<th>Right Bath</th>
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<tbody>
<tr>
<td>0.001 cm</td>
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<td>0.01 cm</td>
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<tr>
<td>1 cm</td>
<td>1 cm</td>
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</tbody>
</table>

a. Assess the validity of assumption 4.

i. Make rough estimates of both the steady-state ($\tau_{ss}$) and equilibrium ($\tau_{eq}$) time constants from the computations.

ii. Estimate the same two time constants based on theoretical considerations (see chapter 3 of volume 1 of the text).

iii. What is your conclusion based on your computations and your estimates of time constants?

b. Assess the validity of assumption 3.

i. Before you do the computation, make an estimate of the nal concentration in each bath. Then do the computation, and check your initial estimates against the computed values.

ii. If they differ, explain the basis of the difference.

iii. How good is the assumption that the quantity of solute in the membrane is negligible? If you decide that the quantity of solute in the membrane is not negligible, design a simulation experiment to test your conclusion.

e. Are the bath concentrations exponential functions of time? Explain.