Linear transformations of random vectors: \( \vec{Y} = r(\vec{X}) \)

\[
\begin{vmatrix}
  y_1 \\
  . \\
  . \\
  . \\
  y_n
\end{vmatrix}
= \begin{vmatrix}
  x_1 \\
  . \\
  . \\
  . \\
  x_n
\end{vmatrix}
\]

A \( n \times n \) matrix, \( \vec{X} = A^{-1}\vec{Y} \) if \( \det A \neq 0 \rightarrow A^{-1} = B \)

where \( \vec{X} = x_1, x_2 \) with p.d.f.:

\[
f(x_1, x_2) = \begin{cases} \{c x_1 x_2, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1; 0 \text{ otherwise} \} 
\end{cases}
\]

To make integral equal 1, \( c = 4 \).

New joint function:

\[
Y_1 = X_1 + 2X_2, Y_2 = 2X_1 + X_2; A = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \rightarrow \det(A) = -3
\]

Calculate the inverse functions:

\[
X_1 = -\frac{1}{3}(Y_1 - 2Y_2), X_2 = -\frac{1}{3}(Y_2 - 2Y_1)
\]

New joint function:

\[
g(y_1, y_2) = \begin{cases} \frac{1}{3} \times 4(-\frac{1}{3}(y_1 - 2y_2))(-\frac{1}{3}(y_2 - 2y_1)) & 
\end{cases}
\]

for \( -\frac{1}{3}(y_1 - 2y_2) \leq 1 \) and \( -\frac{1}{3}(y_2 - 2y_1) \leq 1; \)

\[
0, \text{ otherwise}
\]

Simplified:

\[
f(y_1, y_2) = \begin{cases} \frac{4}{27}(y_1 - 2y_2)(y_2 - 2y_1) & 
\end{cases}
\]

for \( -3 \leq y_1 - 2y_2 \leq 0, -3 \leq y_2 - 2y_1 \leq 0; \)

\[
0, \text{ otherwise}
\]
Linear transformation distorts the graph from a square to a parallelogram.

Note: From Lecture 13, when min() and max() functions were introduced, such functions describe engines in series (min) and parallel (max).
When in series, the length of time a device will function is equal to the minimum life in all the engines (weakest link).
When in parallel, this is avoided as a device can function as long as one engine functions.

**Review of Problems from PSet 4 for the upcoming exam:** (see solutions for more details)

Problem 1 - \( f(x) = \{ ce^{-2x} \text{ for } x \geq 0; 0 \text{ otherwise} \} \)
Find \( c \) by integrating over the range and setting equal to 1:

\[
1 = \int_{0}^{\infty} ce^{-2x} \, dx = -\frac{1}{2}ce^{-2x}\bigg|_{0}^{\infty} = -\frac{c}{2} \times -1 = 1 \rightarrow c = 2
\]

\( \mathbb{P}(1 \leq X \leq 2) = \int_{1}^{2} 2e^{-2x} \, dx = e^{-2} - e^{-4} \)

Problem 3 - \( X \sim U[0, 5], Y = 0 \text{ if } X \leq 1; Y = X \text{ if } 1 \leq X \leq 3; Y = 5 \text{ if } 3 < X \leq 5 \)
Draw the c.d.f. of \( Y \), showing \( \mathbb{P}(Y \leq y) \)

Graph of \( Y \) vs. \( X \), not the c.d.f.

Write in terms of \( X \rightarrow \mathbb{P}(X - ?) \)

Cumulative Distribution Function
Cases:
y < 0 → \Pr(Y ≤ y) = \Pr(0) = 0
0 ≤ y ≤ 1 → \Pr(Y ≤ y) = \Pr(0 ≤ X ≤ 1) = 1/5
1 < y ≤ 3 → \Pr(Y ≤ y) = \Pr(X ≤ y) = y/5
3 < y ≤ 5 → \Pr(Y ≤ y) = \Pr(X ≤ 3) = 3/5
y > 5 → \Pr(Y ≤ 5) = \Pr(X ≥ 5) = 1
These values over X from 0 to \infty give its c.d.f.

Problem 8 - 0 ≤ x ≤ 3, 0 ≤ y ≤ 4
c.d.f. \( F(x, y) = \frac{1}{156}xy(x^2 + y) \)
\( \Pr(1 ≤ x ≤ 2, 1 ≤ y ≤ 2) = F(2, 2) - F(2, 1) - F(1, 2) + F(1, 1) \)

or, you can find the p.d.f. and integrate (more complicated):
c.d.f. of Y: \( \Pr(Y ≤ y) = \Pr(X ≤ \infty, Y ≤ y) = \Pr(X ≤ 3, Y ≤ y) \)
(based on the domain of the joint c.d.f.)
\( \Pr(Y ≤ y) = \frac{1}{156}3y(9 + y) \) for 0 ≤ y ≤ 4
Must also mention: y ≤ 0, \( \Pr(Y ≤ y) = 0; y ≥ 4, \Pr(Y ≤ y) = 1 \)
Find the joint p.d.f. of x and y:
\[ f(x, y) = \frac{∂^2 F(x, y)}{∂x∂y} = \begin{cases} \frac{1}{156}(3x^2 + 2y), & 0 ≤ x ≤ 3, 0 ≤ y ≤ 4; \\ 0 \text{ otherwise} \end{cases} \]

\[ \Pr(Y ≤ X) = \int_{y≤x} f(x, y)dxdy = \int_{0}^{3} \int_{0}^{x} \frac{1}{156}(3x^2 + 2y)dydx = \frac{93}{208} \]

** End of Lecture 14**