Review for Exam 2

pg. 280, Problem 5
\[ \mu = 300, \sigma = 10; X_1, X_2, X_3 \sim N(300, 100) = \sigma^2 \]
\[ P(X_1 > 290 \cup X_2 > 290 \cup X_3 > 290) = 1 - P(X_1 \leq 290)P(X_2 \leq 290)P(X_3 \leq 290) \]
\[ = 1 - P\left( X_1 - \frac{300}{10} \leq -1 \right)P\left( X_2 - \frac{300}{10} \leq -1 \right)P\left( X_3 - \frac{300}{10} \leq -1 \right) \]
Table for x = 1 gives 0.8413, x = -1 is therefore 1 - 0.8413 = 0.1587
\[ = 1 - (0.1587)^3 = 0.996 \]

pg. 291, Problem 11
600 seniors, a third bring both parents, a third bring 1 parent, a third bring no parents.
Find \( P(< 650 \text{ parents}) \)
\[ X_i = 0, 1, 2 \rightarrow \text{parents for the } i\text{th student.} \]
\[ P(X_i = 2) = P(X_i = 1) = P(X_i = 0) = \frac{1}{3} \]
\[ P(X_1 + \ldots + X_{600} < 650) \text{ - use central limit theorem.} \]
\[ \mu = 0(1/3) + 1(1/3) + 2(1/3) = 1 \]
\[ \mathbb{E}X^2 = 0^2(1/3) + 1^2(1/3) + 2^2(1/3) = \frac{5}{3} \]
\[ \text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{5}{3} - 1 = \frac{2}{3} \]
\[ \sigma = \sqrt{2/3} \]
\[ \frac{P\left( \sqrt{600\left( \frac{\sum x_i}{600} - 1 \right)} < \frac{(650 - 1)\sqrt{600}}{\sqrt{2/3}} \right)}{P\left( \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} < 2.5 \right)} < 2.5 \approx N(0, 1), P(Z \leq 2.5) = \phi(2.5) = 0.9938 \]

pg. 354, Problem 10
Time to serve \( X \sim E(\theta), n = 20, X_1, \ldots, X_{20}, \bar{X} = 3.8 \text{ min} \)
Prior distribution of \( \theta \) is a Gamma dist. with mean 0.2 and std. dev. 1
\( \alpha/\beta = 0.2, \alpha/\beta^2 = 1 \rightarrow \beta = 0.2, \alpha = 0.04 \)
Get the posterior distribution:
\[ f(x|\theta) = \theta e^{-\theta x}, f(x_1, \ldots, x_n|\theta) = \theta^n e^{-\theta \sum x_i} \]
\[ f(\theta) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \theta^{\alpha-1} e^{-\theta \sum x_i} \]
Posterior is \( \Gamma(\alpha + n, \beta + \sum x_i) = \Gamma(0.04 + 20, 0.2 + 3.8(20)) \)
Bayes estimator = mean of posterior distribution =
\[ = \frac{20.04}{3.8(20) + 0.2} \]

Problem 4
\[ f(x|\theta) = \{e^{\theta x}, x \geq 0; 0, x < 0\} \]
Find the MLE of \( \theta \)
Likelihood \( \phi(\theta) = f(x_1|\theta) \times \ldots \times f(x_n|\theta) \)
\[ = e^{\theta x_1} \ldots e^{\theta x_n} I(x_1 \geq \theta, \ldots, x_n \geq \theta) = e^{n\theta - \sum x_i} I(\min(x_1, \ldots, x_n) \geq \theta) \]
Maximize over $\theta$.

Note that the graph increases in $\theta$, but $\theta$ must be less than the min value. If greater, the value drops to zero. Therefore:

$$\hat{\theta} = \min(x_1, \ldots, x_n)$$

Also, by observing the original distribution, the maximum probability is at the smallest $X_i$.

p. 415, Problem 7:
To get the confidence interval, compute the average and sample variances:
Confidence interval for $\mu$:

$$\bar{x} - c \sqrt{\frac{1}{n-1}(\bar{x}^2 - \overline{(x^2)})} \leq \mu \leq \bar{x} - c \sqrt{\frac{1}{n-1}(\bar{x}^2 - \overline{(x^2)})}$$

To find $c$, use the t distribution with $n - 1$ degrees of freedom:

$$t_{n-1} = t_{19(-\infty, c)} = 0.95, c = 1.729$$

Confidence interval for $\sigma^2$:

$$\frac{\sqrt{n(x - \mu)}}{\sigma} \sim N(0, 1), \frac{n(x^2 - \overline{(x^2)})}{\sigma^2} \sim \chi^2_{n-1}$$
\[
\frac{t_{n-1}}{\sqrt{\frac{1}{n-1} \chi^2_{n-1}}} = \sqrt{n(\bar{x} - \mu)/\sigma} \sim t_{n-1}
\]

Use the table for \( \chi^2_{n-1} \)

From the Practice Problems:
(see solutions for more detail)

p. 196, Number 9
\[ P(X_1 = \text{defective}) = p \]
Find \( E(X - Y) \)
\( X_i = \{1, \text{defective}; -1, \text{not defective}\}; X - Y = X_1 + \ldots + X_n \)
\[ E(X - Y) = E X_1 + \ldots + E X_n = n E X_1 = n(1 \times p - 1(1 - P)) = n(2p - 1) \]

p. 396, Number 10
\( X_1, \ldots, X_6 \sim N(0, 1) \)
\[ c((X_1 + X_2 + X_3)^2 + (X_4 + X_5 + X_6)^2) \sim \chi^2_n \]
\[ (\sqrt{c}(X_1 + X_2 + X_3))^2 + (\sqrt{c}(X_4 + X_5 + X_6))^2 \sim \chi^2_2 \]
But each needs a distribution of \( N(0, 1) \)
\[ E\sqrt{c}(X_1 + X_2 + X_3) = \sqrt{E X_1 + E X_2 + E X_3} = 0 \]
\[ \text{Var}(\sqrt{c}(X_1 + X_2 + X_3)) = c(\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3)) = 3c \]
In order to have the standard normal distribution, variance must equal 1.
\[ 3c = 1, c = 1/3 \]

** End of Lecture 28**