Take sample $X_1, \ldots, X_n \sim N(0, 1)$

$$A = \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \sim N(0, 1), B = \frac{n(\bar{x}^2 - (\bar{x})^2)}{\sigma^2} \sim \chi^2_{n-1}$$

A, B - independent.

To determine the confidence interval for $\mu$, must eliminate $\sigma$ from A:

$$\frac{A}{\sqrt{\frac{1}{n-1}B}} = \frac{Z^0}{\sqrt{\frac{1}{n-1}(z_1^2 + \ldots + z_{n-1}^2)}} \sim t_{n-1}$$

Where $Z_0, Z_1, \ldots, Z_{n-1} \sim N(0, 1)$

The standard normal is a symmetric distribution, and $\frac{1}{n-1}(Z_1^2 + \ldots + Z_{n-1}^2) \to \mathbb{E}Z_1^2 = 1$

So $t_{n}$-distribution still looks like a normal distribution (especially for large $n$), and it is symmetric about zero.

Given $\alpha \in (0, 1)$ find $c$, $t_{n-1}(-c, c) = \alpha$

$$-c \leq \frac{A}{\sqrt{\frac{1}{n-1}B}} \leq c$$

with probability = confidence ($\alpha$)

$$-c \leq \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \sqrt{\frac{1}{n-1}\frac{n(\bar{x}^2 - (\bar{x})^2)}{\sigma^2}} \leq c$$

$$-c \leq \frac{\bar{x} - \mu}{\sqrt{\frac{1}{n-1}(\bar{x}^2 - (\bar{x})^2)}} \leq c$$

$$\bar{x} - c\sqrt{\frac{1}{n-1}(\bar{x}^2 - (\bar{x})^2)} \leq \mu \leq \bar{x} + c\sqrt{\frac{1}{n-1}(\bar{x}^2 - (\bar{x})^2)}$$

By the law of large numbers, $\bar{x} \to \mathbb{E}X = \mu$

The center of the interval is a typical estimator (for example, MLE).

error $\propto$ estimate of variance $\approx \sqrt{\frac{s^2}{n}}$ for large $n$.

$s^2 = \bar{x}^2 - (\bar{x})^2$ is a sample variance and it converges to the true variance,
by LLN \( \sigma^2 \to \sigma^2 \)

\[
\mathbb{E} \hat{\sigma}^2 = \mathbb{E} \frac{1}{n}(x_1^2 + \ldots + x_n^2) - \mathbb{E}\left(\frac{1}{n}(x_1 + \ldots + x_n)^2\right) = \\
= \mathbb{E}X_i^2 - \frac{1}{n^2} \sum_{i,j} \mathbb{E}X_iX_j = \mathbb{E}X_1^2 - \frac{1}{n^2}(n\mathbb{E}X_1^2 + n(n+1)(\mathbb{E}X_1)^2)
\]

Note that for \( i \neq j, \mathbb{E}X_iX_j = \mathbb{E}X_i\mathbb{E}X_j = (\mathbb{E}X_1)^2 = \mu^2, \ n(n-1) \) terms with different indices.

\[
\mathbb{E} \hat{\sigma}^2 = \frac{n-1}{n} \mathbb{E}X_1^2 - \frac{n-1}{n} (\mathbb{E}X_1)^2 = \\
= \frac{n-1}{n} (\mathbb{E}X_1^2 - (\mathbb{E}X_1)^2) = \frac{n-1}{n} \text{Var}(X_1) = \frac{n-1}{n}\sigma^2
\]

Therefore:

\[
\mathbb{E} \hat{\sigma}^2 = \frac{n-1}{n}\sigma^2 < \sigma^2
\]

Good estimator, but more often than not, less than actual.
So, to compensate for the lower error:

\[
\mathbb{E} \frac{n}{n-1} \hat{\sigma}^2 = \sigma^2
\]

Consider \((\sigma')^2 = \frac{n}{n-1}\hat{\sigma}^2\), unbiased sample variance.

\[
\pm c \sqrt{\frac{1}{n-1}(\bar{x}^2 - (\bar{x})^2)} = \pm c \sqrt{\frac{1}{n-1} \hat{\sigma}^2} = \pm c \sqrt{\frac{1}{n}(\sigma')^2}
\]

\[
\bar{x} - c \sqrt{\frac{(\sigma')^2}{n}} \leq \mu \leq \bar{x} + c \sqrt{\frac{(\sigma')^2}{n}}
\]

§7.5 pg. 140 Example: Lactic Acid in Cheese
0.86, 1.53, 1.57, ..., 1.58, \( n = 10 \)
\sim N(\mu, \sigma^2), \bar{x} = 1.379, \hat{\sigma}^2 = \bar{x}^2 - (\bar{x})^2 = 0.0966
Predict parameters with confidence \( \alpha = 95\% \)
Use a t-distribution with \( n - 1 = 9 \) degrees of freedom.
See table: $(-\infty, c) = 0.975$ gives $c = 2.262$

$$x - 2.262\sqrt{\frac{1}{2}\sigma^2} \leq \mu \leq x + 2.262\sqrt{\frac{1}{2}\sigma^2}$$

$$0.6377 \leq \mu \leq 2.1203$$

Large interval due to a high guarantee and a small number of samples.
If we change $\alpha$ to 90% $\ c = 1.833$, interval: $1.189 \leq \mu \leq 1.569$
Much better sized interval.

Confidence interval for variance:

$$c_1 \leq \frac{n\hat{\sigma}^2}{\sigma^2} \leq c_2$$

where the $c$ values come from the $\chi^2$ distribution

$$\chi^2$$

Not symmetric, all positive points given for $\chi^2$ distribution.
$c_1 = 2.7$, $c_2 = 19.02 \to 0.0508 \leq \sigma^2 \leq 0.3579$
again, wide interval as result of small $n$ and high confidence.

**Sketch of Fisher’s theorem.**

$z_1, \ldots, z_n \sim N(0, 1)$

$$\sqrt{n} \overline{z} = \frac{1}{\sqrt{n}}(z_1 + \ldots + z_n) \sim N(0, 1)$$

$$n (\overline{z}^2 - (\overline{z})^2) = n (\frac{1}{n} \sum z_i^2 - (\frac{1}{n} \sum z_i)^2) = \sum z_i^2 - (\frac{1}{\sqrt{n}}(z_1 + \ldots + z_n))^2 \sim \chi^2_{n-1}$$

$$f(z_1, \ldots, z_n) = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2 \sum z_i^2} = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2 \overline{z}^2}$$

$$f(y_1, \ldots, y_n) = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2 \sum y_i^2} = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2 \overline{y}^2}$$

The graph is symmetric with respect to rotation, so rotating the coordinates gives again i.i.d. standard normal sequence.

$$\prod_i \frac{1}{\sqrt{2\pi}} e^{-1/2 \overline{y}_i^2} \to y_1, \ldots, y_n - i.i.d.N(0, 1)$$

Choose coordinate system such that:

$$y_1 = \frac{1}{\sqrt{n}}(z_1 + \ldots + z_n), \text{ i.e. } \overline{v}_1 = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) - \text{new first axis.}$$

Choose all other vectors however you want to make a new orthogonal basis:
\[ y_1^2 + \ldots + y_n^2 = z_1^2 + \ldots + z_n^2 \]
since the length does not change after rotation!

\[ \sqrt{n}z = y_1 \sim N(0,1) \]

\[ n(z^2 - \left(\overline{z}\right)^2) = \sum y_i^2 - y_1^2 = y_2^2 + \ldots + y_n^2 \sim \chi^2_{n-1} \]

** End of Lecture 27 **