

Problem Set 5

Part A

From the Discrete to Continuum Modeling Notes (March 18), page 15, section 4, do problems: 5, 6, 8 and 9.

Part B: The Freezing Lake

A shallow (10 cm deep) body of water initially at 10°C is exposed to air at -10°C, and begins to freeze from the top. Here we will use the finite difference method to calculate the temperature profile across the liquid water and ice, and model the growth of the ice as well as the heat conduction. We will neglect buoyancy instabilities in the water and treat it as stagnant, and also neglect the small change in overall thickness due to the lower density of ice.

Properties:

Material	Liq. water	Sol. water
Conductivity k , $\frac{\text{W}}{\text{m}\cdot\text{K}}$	0.56	2.3
Heat capacity c_p , $\frac{\text{J}}{\text{kg}\cdot\text{K}}$	4200	2100
Density ρ , $\frac{\text{kg}}{\text{m}^3}$	1000	920
Heat of fusion ΔH_f , $\frac{\text{J}}{\text{kg}}$	334,000	

Problems:

1. Write the difference equation corresponding to the following differential equation at an internal node using the forward Euler algorithm:

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) = 0$$

You can “close” this equation in your finite difference calculation by creating separate arrays to hold enthalpy and temperature, so $T_{i,n}$ is calculated from $H_{i,n}$, and $H_{i,n+1}$ is calculated from $T_{i,n}$ and its spatial neighbors. (6)

2. At the bottom of the lake ($x = 10\text{cm}$), we will assume that the lake bed is a perfect insulator, so the boundary condition is equivalent to a *symmetry plane*. At the top surface ($x = 0$), the heat flux is given by a heat transfer coefficient: $q_x = h(T_{\text{air}} - T_{\text{water}})$.

Write the difference equations for the top and bottom finite difference nodes in your system.

(6)

3. Calculate the thermal diffusivities of liquid and solid water. Which one will determine the largest stable timestep size using the forward Euler algorithm? (2)
4. Estimate the time required to reach steady-state by conduction through 10 cm of liquid water. Estimate the time required for conduction-limited freezing of the whole lake, assuming ice is at the air temperature at the top surface and temperature profile across the ice is linear. (4)

[For the latter, you can model the freezing process by setting the heat flux required to freeze at a certain rate equal to the heat flux due to conduction:

$$\rho\Delta H_f \frac{dX}{dt} = k \frac{\Delta T}{X}$$

and solve the resulting differential equation to determine thickness frozen X vs. time t .]

5. Using at least ten nodes in the x -direction, perform a finite difference calculation to predict the temperature profiles throughout the freezing process with a top heat transfer coefficient of $10 \frac{\text{W}}{\text{m}^2 \cdot \text{K}}$. Provide plots at the times when the top of the lake just starts to freeze, when the lake is half frozen, and when the lake has just finished freezing. (15)

Note: if you use a spreadsheet, the **MIN**, **MAX** and **SIGN** functions may be useful for establishing the $T(H)$ and $k(T)$ functions.

[Worth 1/3 of the total grade for this assignment, so the points add up to 33.]