Bose condensation

1. Quasiparticles.

Consider a Bose gas at $T = 0$ with one quasiparticle with momentum $p \neq 0$ added on the top. Quasiparticle state can be obtained by applying the quasiparticle creation operator to the nonideal Bose gas ground state:

$$|1_p\rangle = \hat{b}_p^+ |0\rangle$$

where $\hat{b}_p^+ = \cosh \theta_p \hat{a}_p^+ - \sinh \theta_p \hat{a}_p$.

How many particles are contained in one quasiparticle? To find out, take the number operator $\hat{N} = \sum_k \hat{a}_k^+ \hat{a}_k$ of the original particles and evaluate the difference

$$\langle \hat{N}_p \rangle = \langle 1_p | \hat{N} | 1_p \rangle - \langle 0 | \hat{N} | 0 \rangle = 0$$

(Be careful: $\hat{b}_p^+ |0\rangle \neq 0$, instead $\hat{b}_p^+ |0\rangle = 0$.)

Express the answer in terms of the Bogoliubov angle $\theta_p$.

Compare the situation at high and low quasiparticle energy and interpret the result.

2. Landau criterion for superfluidity.

Landau 超流體準則

A superflow state of a Bose condensate having velocity $v$ is characterized by macroscopic occupancy of state with nonzero momentum $p = mv$. The many body state can be constructed by generalizing the scheme used to describe stationary condensates:

| $\Phi_v$ $\rangle = \exp(\sqrt{V} (\phi(x) \hat{a}_p - \bar{\phi}(x) \hat{a}_p^+))$, $\phi(x) = \phi \exp\left(\frac{i}{\hbar} px\right)$

(a) Starting from this state, consider the expectation value $\langle \Phi_v | H - \mu \hat{N} | \Phi_v \rangle$ and, by minimizing energy in $\phi$, obtain the chemical potential $\mu$ of the superflow state. How does $\mu$ depend on the superflow velocity $v$?
b) Consider elementary excitations (quasiparticles) in the superflow state. The Bose gas hamiltonian expanded up to second order in $a_k, a_k^*$, has the form

$$H = E_0 + \sum_{k \neq 0} \left( \varepsilon_k^{(0)} - \mu + \lambda |\phi|^2 \right) a_k a_k^* + \frac{1}{2} \sum_{k \neq 0} (\phi^2 a_k^* a_{-p-k}^* + \bar{\phi}^2 a_k a_{-2p-k}^*)$$

(4)

To diagonalize this hamiltonian, group together the states with momenta $k$ and $2p - k$.

Find the parameters $\theta_k$ that diagonalize the Hamiltonian, and obtain the quasiparticle dispersion relation $E(k)$. (Hint: Don’t let yourself be dragged into long calculation, the result can be more or less read off the solution for stationary BEC with slight adjustments.)

Find the critical superflow velocity $\psi_c$ above which the energy of quasiparticles $E(k)$ can become negative. Landau argued that the superfluid can sustain nondissipative flows with velocities $\psi < \psi_c$, and in this way he could explain the phenomenon of critical velocity observed in superfluid $^4$He. At $E(k) > 0$ the quasiparticles cannot be created spontaneously, while at $\psi > \psi_c$ the flow is accompanied by massive quasiparticle creation, and is thus dissipative. Find the critical velocity $\psi_c$ for nonideal Bose gas.

c) Can you interpret the result of part b) for quasiparticle dispersion in superflow from the point of view of a Galilean transformation? Note that the microscopic Hamiltonian is invariant with respect to changing the reference frame from stationary to moving, $x' = x + vt$, $t' = t$. Show that for an excitation with frequency $\omega$ and wavevector $k$ this yields $\omega' = \omega - kv, k' = k$. How is the quasiparticle energy changed under a Galilean transformation?

3. Condensate depletion.

a) In a nonideal Bose gas at $T = 0$ only a fraction of all the particles is found in the condensate. The reduction of condensate density due to interactions is called “condensate depletion.” (An extreme example is provided by $^4$He, where the majority of the particles — more than 90% — are not in the condensate. To estimate this effect in a weakly nonideal Bose gas, find the expectation value of the total density.)
在 $T=0$ 時非理想玻色氣體中，只有一部分的粒子發生凝聚。凝聚密度的減少是由於稱為“凝聚損耗”的作用而產生的（一個極端的例子是 $^4$He，其中大部分粒子（超過90%）不發生凝聚）。

為了評估弱非理想玻色氣體的效應，找出基態密度的期望值

$$\hat{n} = \hat{n}_0 + V^{-1} \sum_{k \neq 0} \hat{n}_k, \quad \hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$$

(6)

over the ground state. The first term gives the condensate density $n_0 = \langle a_0^\dagger a_0 \rangle$, while the second term gives the density of the out-of-condensate particles. Find the depletion factor $(n-n_0)/n$ dependence on the coupling constant $\lambda$.

超過基態。第一項給出凝聚密度 $n_0 = \langle a_0^\dagger a_0 \rangle$，而第二項給出非凝聚粒子的密度。找出依賴於耦合常數 $\lambda$ 的損耗因數 $(n-n_0)/n$。

b) Consider the correlator of the field operators $R(x,x') = \langle 0| \hat{\phi}^+(x) \hat{\phi}(x') |0 \rangle$. Show that it is related to the particle number distribution as $R(x-x') = \sum_k \langle 0| \hat{n}_k |0 \rangle e^{ik(x-x')}$. **Describe the behavior of $R(x-x')$ as a function of point separation** $x-x'$. Find the limits as $|x-x'| \to \infty$ and at $x-x'$. Estimate the length scale $\xi$, called *BEC healing length*, at which the crossover from $R(0)$ to $R(\infty)$ takes place.

考慮場算子 $R(x,x') = \langle 0| \hat{\phi}^+(x) \hat{\phi}(x') |0 \rangle$ 的相關，證明它與粒子數分佈

$$R(x-x') = \sum_k \langle 0| \hat{n}_k |0 \rangle e^{ik(x-x')}$$

有關。描述與點間距 $x-x'$ 有關的 $R(x-x')$ 的行為。找出 $x-x'$ 處的極限，當 $|x-x'| \to \infty$。估算長度因數 $\xi$ （被稱為 *BEC恢復長度*），此處發生 $R(0)$ 到 $R(\infty)$ 的交疊。

4. Thermodynamics of a Bose gas

玻色氣體熱力學

Thermodynamics quantities of Bose-condensed gas can be found by treating the system as a gas of noninteracting Bogoliubov quasiparticles obeying Bose statistics. The thermodynamic potential of the system is

將系統當作服從玻色統計的無相互作用的 Bogoliubov 粒子，則可求得玻色凝聚氣體的熱力學。系

統的熱力學勢是，

$$\Omega = -T \ln Z = T \int \ln [1 - e^{-\beta E(k)}] \frac{d^3 k}{(2\pi)^3}, \quad E(k) = \sqrt{\varepsilon^{(0)}(k)(\varepsilon^{(0)}(k) + 2\lambda n)}$$

(7)

with $\varepsilon^{(0)}(k) = h^2 k^2 / 2m$.

其中，$\varepsilon^{(0)}(k) = h^2 k^2 / 2m$

a) Show that simple analytical results for the thermodynamic potential $\Omega$ can be obtained at Very low temperatures, $T << T_s \approx \hbar n$ and at moderately high temperatures, $T_s << T \leq T_{BEC}$. (Hint: Given the temperature, low or high, simplify the form of $E(k)$ by ignoring $\varepsilon^{(0)}(k)$ compared to $\hbar n$, or vice versa.)

證明熱力學勢 $\Omega$ 的簡單解析結果可以在極低溫， $T << T_s \approx \hbar n$ 及適度高溫 $T_s << T \leq T_{BEC}$ 下得到。（提示：考慮到溫度，高溫或低溫，忽略與 $\hbar n$ 可比擬的 $\varepsilon^{(0)}(k)$ 以簡化 $E(k)$ 的形式，反之亦然）
b) Find the entropy, the specific heat, and the normal component density $n(T)$ in the above two temperature intervals. Compare with the ideal Bose gas.

找出熵，比熱和上述兩個溫度區間下的正常成分密度 $n(T)$，並與理想玻色氣體作比較。