1. Operator identities.

Here we prove two useful theorems from operator algebra that will be used in the problems of this homework and later in the course.

a) Let $\hat{A}$ and $\hat{B}$ be two operator that do not necessarily commute. Prove the so-called operator expansion theorems:

$$\exp(\hat{A}) \exp(-\hat{A}) = \hat{B} + x[\hat{A}, \hat{B}] + \frac{x^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots$$

with $x$ a parameter. (Hint: consider the derivative $f'(x)$ and compare its Taylor series in $x$ with that for $f(x)$.)

In the special case when the commutator $[\hat{A}, \hat{B}] = c$ is a c-number, the series terminates after the second term, giving

$$\exp(x\hat{A}) \exp(-x\hat{A}) = \hat{B} + cx$$

Apply this result to the coordinate and momentum operators, $\hat{B} = \hat{q}, \hat{A} = \hat{p}/h = -i\hbar / dq$.

b) Let $A$ and $B$ be two operator whose commutator $[\hat{A}, \hat{B}]$ commutes with both $A$ and $B$ (e.g., $[\hat{A}, \hat{B}] = c$ a c-number). Prove the Campbell-Baker-Hausdorf theorem:

$$\exp(x\hat{A}) \exp(x\hat{B}) = \exp(x\hat{A} + \hat{B})$$

For that, consider $\hat{C}(x) = \exp(x\hat{A})\exp(x\hat{B})$, differentiate both sides with respect to $x$ and, using the operator expansion theorem(1), show that $d\hat{C}(x)/dx = (\hat{A} + \hat{B} + x[\hat{A}, \hat{B}])\hat{C}(x)$. Integrate with respect to $x$ like an ordinary differential equation.

2. Displacement operators.

a) Consider the displacement operators, defined as

$$\exp(x\hat{A}) \exp(-x\hat{A}) = \hat{B} + cx$$
\[ \hat{D}(v) = \exp(v\hat{a}^* - \hat{a}v) \]  \hspace{1cm} (4)

with \( v \) a complex parameter. Prove unitarity: \( \hat{D} + (v)\hat{D}(v) = 1, \hat{D}^{-1}(v) = \hat{D}(-v). \)

其中 \( v \) 是一复参数。证明幺正性: \( \hat{D} + (v)\hat{D}(v) = 1, \hat{D}^{-1}(v) = \hat{D}(-v). \)

For a real-valued \( v \) show that in the \( q \)-representation the displacement operator (4) acts as an Argument shift:

对实值 \( v \), 证明在 \( q \)-表象中位移算子(4) 引起辐角位移

\[ \hat{D}(v)\psi(q) = \exp(\sqrt{\lambda}q)\psi(q) = \psi(q + \sqrt{2}v) \]

with the length \( \lambda = \sqrt{\hbar/mw} \). (Hint: relate \( \hat{D}(v) \) to the Taylor series formula.)

其中，长度 \( \lambda = \sqrt{\hbar/mw} \)。（提示：关联 \( \hat{D}(v) \) 及 Taylor 级数方程）

b) Show that the coherent states can be obtained by “displacing” the vacuum state, \( |\psi\rangle = \hat{D}(v)|0\rangle \).

(Use the operator expansion theorem(1)).

c) Show that the unitary transformation \( \hat{D}(v) \) displace \( \hat{a} \) by \( v \), and \( \hat{a}^* \) by \( \sqrt{2}v \),

证明幺正变换 \( \hat{D}(v) \) 使 \( \hat{a} \) 位移 \( v \), 及使 \( \hat{a}^* \) 位移 \( \sqrt{2}v \),

\[ \hat{D}^+(v)\hat{a}\hat{D}(v) = \hat{a} + v \]
\[ \hat{D}^+(v)\hat{a}^*\hat{D}(v) = \hat{a}^* + \sqrt{2}v \]  \hspace{1cm} (6)
For any function of operation $\hat{a}$ and $\hat{a}^+$ with a power series expansion, show that

$$\hat{D}^+(v)f(\hat{a},\hat{a}^+)\hat{D}(v) = f(\hat{a} + v,\hat{a}^+ + \overline{v})$$  (7)

d) Prove the product formula

$$\hat{D}(v')\hat{D}(v) = e^{iv - iv'}\hat{D}(v + v')$$  (8)

Note that the displacement operators $\hat{D}(v)$ and $\hat{D}(v')$ commute only when $\arg(v) = \arg(v')$.

3. Harmonic oscillator excited by an external force.

In the presence of a time-dependent force, $H = \frac{1}{2}\hbar\omega(\hat{p}^2 + \hat{q}^2) - F(t)\hat{q}$. Show that the evolution in time of an arbitrary coherent state can be obtained using the displacement operators (4) studied in Problem 2. Assume that the evolved coherent state remains a coherent state at all times, so that

$$|\alpha\rangle(t) = \hat{D}(v(t))|\alpha\rangle = |\alpha + v(t)\rangle$$  (9)

Obtain a differential equation for the function $v(t)$ and show that its real and imaginary parts correspond to the classical Hamilton equations $dq/dt = p, dp/dt = F(t)$.

Show that the unitary transformation $H' = \hat{D}(v(t))HD^{-1}(v(t))$ gives a free oscillator Hamiltonian with $F(t) = 0$. It describes the transformation of the quantum problem to the classical co-moving reference frame.

How does the function $v(t)$ should evolve in time in order for it such that at all times it remains a coherent state

b) The harmonic oscillator of part a), initially in the ground state, was subject to a constant force during the time interval $0 < t < \tau$. Find the state at $t > \tau$. Determine the distribution of energies. Determine the distribution of energies.

c) For the state found in part b) at $t > \tau$, find the phase-space density, i.e., the Wigner function $W(q,p)$, as a function of time.

d) For the state found in part b) at $t > \tau$, find the phase-space density, i.e., the Wigner function $W(q,p)$, as a function of time.